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THE FRACTURE OF A PHYSICALLY NON-LINEAR INHOMOGENEOUS MEDIUM UNDER STRESS–RELAXATION CONDITIONS IN INCLUSIONS†

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An isotropic elastic plane is considered which contains different elliptic inclusions remote from one another and which exhibit the properties of non-linear creep. The corresponding constitutive equations contain a damage parameter which varies from zero (in the undeformed state) to unity (at the instant of fracture). Loads which are constant in time act at infinity which cause the relaxation of stresses in the inclusions. The conditions are obtained under which: (a) fracture of inclusions occurs, and (b) fracture is impossible. The results are generalized to the case of a finite domain with a non-linear inclusion of arbitrary form which is under relaxation conditions in a homogeneous stress–strain state.

An isotropic elastic plane containing an elliptic physically non-linear inclusion (EPNI) and subjected to the action of finite stresses at infinity has been considered previously [1–3]. Relations between the stress–strain states in the inclusion and at infinity were established. In this paper, using these relations, we consider EPNIs which are separated from one another by sufficiently large distances, exhibit non-linear viscous properties (creep) and are damaged during the creep process. In the case of external loads on EPNIs which are constant in time, a relaxation process occurs which is characterized by a fall in the stress level. The question arises of the possibility of fracture (that is, the accumulation of the limit value of the damage parameter) of the inclusions under these conditions. An answer to this is given below.

1. FORMULATION OF THE PROBLEM

We consider an isotropic elastic plane with different elliptic, physically non-linear inclusions (EPNI) which are under conditions of plane strain or a generalized plane stressed state as a consequence of constant external loads acting at infinity. We assume that the distance between the centres of any two arbitrary EPNIs is large compared to their dimensions so that the mutual effect of each inclusion on the stress–strain state of any other inclusion can be neglected. We therefore choose some arbitrarily selected EPNI, which we denote by S^*, and introduce a system of coordinates Ox_1x_2 such that the equation of the boundary L^* which separates S^* from the elastic medium S has the form

$$x_1^2a^{-2} + x_2^2b^{-2} = 1, \quad a \ge b$$

The principal values of the stresses acting at infinity are denoted by N_1 and N_2 and the angle between the first principal axis and the Ox_1 axis is denoted by α.

Hooke's law [1–3]

$$8\mu\varepsilon_{kl} = (\kappa-1)\sigma_{nn}\delta_{kl} + 4\sigma_{kl}^0, \quad \sigma_{kl}^0 = \sigma_{kl} - \sigma_{nn}\delta_{kl}/2, \quad k, l = 1, 2 \tag{1.1}$$

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holds in the domain S, where σ_{kl}^0 and δ_{kl} are components of the stress deviator and the unit tensor, μ is the shear modulus, $\kappa = 3 - 4\nu$ in the case of plane deformation and $\kappa = (3 - \nu)/(1 + \nu)$ in the case of a generalized stressed state, ν is Poisson's ratio and summation from 1 to 2 is carried out with respect to repeated subscripts.

The overall strains in the inclusion S^* consist of elastic strains and creep strains ε_{kl}^{*c}, the velocities of which depend on the stresses σ_{kl}^* and the damage parameter so that the constitutive equations have the form [4, 5]

$$\varepsilon_{kl}^* = a_{klmn}^*\sigma_{mn}^* + \varepsilon_{kl}^{*c}, \quad \dot{\varepsilon}_{kl}^{*c} = B_1 s_*^n(1-\omega)^{-m}\partial s_*/\partial\sigma_{kl}^*, \quad k, l = 1, 2 \tag{1.2}$$

where a_{klmn}^* are the components of the elastic compliances of the inclusion, $s_* = s_*(\sigma_{kl}^*)$ is a homogeneous positive convex function of the first degree, ω $(0 \le \omega \le 1)$ is the damage parameter (in the natural state $\omega = 0$ and, at the instant of fracture, $\omega = 1$) and B_1, B_2, m, n and p are positive constants.

The deformations of the elastic medium and the inclusion are assumed to be small, and the field of the loads and the displacement field on the boundary L^* are neglected. The stresses at infinity are applied at the instant of time $t = 0$ and remain constant. When $t < 0$, the domains S and S^* are in a natural undeformed state and, therefore,

$$\omega|_{t=0} = 0, \quad \varepsilon_{kl}^{*c}\Big|_{t=0} = 0, \quad k, l = 1, 2 \tag{1.3}$$

The relations between the stress–strain state in S^* and at infinity (subject to the condition that the magnitude of the rotation $\varepsilon^\infty = 0$), in the system of coordinates Ox_1x_2 connected with the axes of symmetry of the EPNI, have the form [13]

$$\begin{aligned} \mu(m_0\bar{C} + \bar{D}) &= \kappa(m_0 A + B) - (\kappa + 1)(m_0\Gamma + \Gamma') \\ \mu(\bar{C} + m_0\bar{D}) &= -(A + m_0 B) + (\kappa + 1)\Gamma \end{aligned} \tag{1.4}$$

$$2A = \sigma_{11}^* + \sigma_{22}^*, \quad 2B = \sigma_{22}^* - \sigma_{11}^* + 2i\sigma_{12}^*, \quad C = \varepsilon_{11}^* + \varepsilon_{22}^* + 2i\varepsilon^*$$

$$D = \varepsilon_{11}^* - \varepsilon_{22}^* + 2i\varepsilon_{12}^*, \quad m_0 = (a-b)/(a+b) \quad (0 \le m_0 < 1)$$

$$4\Gamma = N_1 + N_2 = \sigma_{11}^\infty + \sigma_{22}^\infty, \quad 2\Gamma' = (N_2 - N_1)\exp(-2i\alpha) = \sigma_{22}^\infty - \sigma_{11}^\infty + 2i\sigma_{12}^\infty$$

where σ_{kl}^∞ $(k, l = 1, 2)$ are the components of the stresses at infinity and ε^* is the magnitude of the rotation in S^*. It was assumed in deriving relations (1.3) that the point (0, 0) was fixed, that is, the displacements of the centre of the EPNI are equal to zero.

Note that the stress–strain state in the EPNI will be homogeneous [1–3].

Equations (1.2) and (1.4) and the initial conditions (1.3) are a closed system for finding the stress–strain state in the inclusion using the known history of the stresses $\sigma_{kl}^\infty = \sigma_{kl}^\infty(t)$ at infinity, that is, $\sigma_{kl}^* = \sigma_{kl}^*(t)$, $\varepsilon_{kl}^* = \varepsilon_{kl}^*(t)$ and $\omega = \omega(t)$ $(k, l = 1, 2)$.

Solving Eq. (1.4) for ε_{kl}^* and ε^*, we find that

$$2\mu(1 - m_0^2)\varepsilon^* = m_0(\kappa + 1)(\sigma_{12}^* - \sigma_{12}^\infty)$$

and, for the remaining three equations, we obtain [2, 3]

$$F_i = \alpha_{ij}y_j + \beta_{ij}\xi_j, \quad i = 1, 2, 3$$

$$F_k = \varepsilon_{kk}^* \quad F_3 = 2\varepsilon_{12}^*, \quad y_k = \sigma_{kk}^*, \quad y_3 = \sigma_{12}^*, \quad \xi_k = \sigma_{kk}^\infty, \quad \xi_3 = \sigma_{12}^\infty; \quad k = 1, 2$$

$$\alpha_{11} = -\frac{(\kappa+1)(1-m_0)}{4\mu(1+m_0)}, \ \alpha_{12} = \alpha_{21} = \frac{\kappa - 1}{4\mu}, \ \alpha_{22} = -\frac{(\kappa+1)(1+m_0)}{4\mu(1-m_0)}, \ \alpha_{33} = -\frac{\kappa + m_0^2}{\mu(1-m_0^2)} \tag{1.5}$$

$$\beta_{11} = \frac{(\kappa+1)(3-m_0)}{8\mu(1+m_0)}, \quad \beta_{12} = \beta_{21} = -\frac{\kappa-1}{8\mu}, \quad \beta_{22} = \frac{(\kappa+1)(3+m_0)}{8\mu(1-m_0)}, \quad \beta_{33} = \frac{\kappa+1}{\mu(1-m_0^2)}$$

$$0 \le m_0 < 1$$

(all of the remaining α_{ij} and β_{ij} are equal to zero; summation with respect to j is carried out from 1 to 3).

Using the notation adopted in (1.5), we write the constitutive equations (1.2) in the form

$$\begin{gathered} F_i = s_e \partial s_e / \partial y_i + f_i^c, \quad f_i^c = B_1 s_c^n (1-\omega)^{-m} \partial s_c / \partial y_i, \quad i = 1,2,3; \quad \dot{\omega} = B_2 s_c^p (1-\omega)^{-m} \\ s_d^2 = \alpha_{ij}^d y_i y_j, \quad f_1^c = \varepsilon_{11}^{*c}, \quad f_2^c = \varepsilon_{22}^{*c}, \quad f_3^c = 2\varepsilon_{12}^{*c}, \quad \alpha_{ij}^d = \alpha_{ji}^d, \quad i,j = 1,2,3; \quad d = e, c \end{gathered} \tag{1.6}$$

The quantities α_{ij}^e are expressed in terms of the components of the elastic compliances a_{klmn}^* from (1.2) and the form of the function s_c in the creep equation in (1.6) is an extension of the relations $s_c = s_c(y_i)$ to the case of an isotropic medium for which $\alpha_{11}^c = \alpha_{22}^c = 1$, $\alpha_{12}^c = 1 - \beta/2$, $\alpha_{33}^c = \beta$. The remaining α_{ij}^c are equal to zero and s_c is identical to the stress intensity (when $\beta = 3$) or to twice the principal shear stress (when $\beta = 4$).

It follows from relations (1.5) and (1.6) that

$$A_{ij} y_j + f_i^c = \beta_{ij} \xi_j, \quad i = 1,2,3; \quad A_{ij} = \alpha_{ij}^e - \alpha_{ij} \tag{1.7}$$

where the matrix $\|A_{ij}\|$ is a symmetric, positive-definite matrix in view of the fact that the matrices $\|\alpha_{ij}^e\|$ and $\|-\alpha_{ij}\|$ are of this type since, according to relations (1.5),

$$\alpha_{11} < 0, \quad \alpha_{11}\alpha_{22} - \alpha_{12}^2 = \kappa\mu^{-2}/4 > 0, \quad \alpha_{33} < 0$$

Since the loads at infinity are constant, that is, $\dot{\xi}_i = 0$ ($i = 1, 2, 3$), then, after differentiating equalities (1.7) with respect to t, multiplying by y_i and summing with respect to i, taking account of Eqs (1.6), we obtain

$$s\dot{s} + B_1 s_c^{n+1} (1-\omega)^{-m} = 0, \quad s(y_i) \equiv (A_{ij} y_i y_j)^{1/2} \tag{1.8}$$

It follows from equalities (1.8) that $\dot{s} < 0$, that is, stress relaxation occurs in the EPNI. The fundamental question is whether it is possible in this relaxation process to accumulate the limit value $\omega = 1$, which corresponds to fracture of the inclusion.

2. THE SUFFICIENT CONDITIONS FOR FRACTURE OF AN EPNI

It has been mentioned above that the matrix $\|A_{ij}\|$ from (1.7) is positive-definite, that is, $A_{ij} y_i y_j > 0$ when $y_i y_i \neq 0$. We shall assume that the matrix $\|\alpha_{ij}^c\|$ from (1.6) also possesses this property: $\alpha_{ij}^c y_i y_j > 0$ when $y_i y_i \neq 0$ (the case of positive semi-definiteness when $\alpha_{ij}^c y_i y_j \geq 0$ if $y_i y_i \neq 0$ will be considered separately in Section 4). It is well known [6] that the numbers $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$, which are the smallest and largest roots of the characteristic equation of a regular sheaf of quadratic forms which have the form $|A_{ij} - \lambda\alpha_{ij}^c| = 0$, will exist such that the following inequalities will hold

$$a_1 s_c \leq s \leq a_2 s_c, \quad a_1 \equiv \sqrt{\lambda_{\min}}, \quad a_2 \equiv \sqrt{\lambda_{\max}} \tag{2.1}$$

The functions $s_c(y_i)$ and $s(y_i)$ are defined in (1.6) and (1.8).

It follows from the third condition of (1.6) that $\omega = \omega(t)$ is an increasing function and therefore, as earlier in [4, 5], we choose it as the independent variable, that is, we shall assume that $y_i = y_i(\omega)$. Then, differentiating equalities (1.7) with respect to ω and taking account of the fact that, according to (1.6)

$$\frac{d}{d\omega} = B_2^{-1} s_c^{-p} (1-\omega)^m \frac{d}{dt}$$

we obtain

$$A_{ij} y_i' + B_1 B_2^{-1} s_c^{n-p} \partial s_c / \partial y_i = 0, \quad i = 1,2,3 \tag{2.2}$$

where the prime denotes differentiation with respect to ω.

Equalities (2.2) form a system of equations for finding the functions $y_i = y_i(\omega)$, the initial conditions for which follow from relations (1.3) and (1.7) and have the form

$$y_{i0} \equiv y_i(0) = B_{ik}\beta_{jk}\xi_j, \quad i = 1, 2, 3 \tag{2.3}$$

where $||B_{ij}||$ is the inverse matrix to $||A_{ij}||$.

Multiplying equalities (2.2) by y_i and summing with respect to i, we find

$$ss' + B_1 B_2^{-1} s_c^{n-p+1} = 0 \tag{2.4}$$

If $s \equiv a_0 s_c$, $a_0 = \text{const}$, then equality (2.4) reduces to the equation for the relaxation of the stress s_c in a rod, which has been investigated earlier in [4]. In the general case, inequalities (2.1) hold, which we shall use in order to obtain the corresponding estimates.

We will consider three cases separately.

1. Suppose $n + 1 - p > 0$. Then, as a consequence of inequalities (2.1), we shall have

$$(s/a_2)^{n+1-p} \le s_c^{n+1-p} \le (s/a_1)^{n+1-p} \tag{2.5}$$

and, from relations (2.4) and (2.5), we obtain

$$\begin{gathered} -A_1 s_0^{p-n+1} \le (s^{p-n+1})'/(p-n+1) \le -A_2 s_0^{p-n+1} \\ A_k \equiv B_1 B_2^{-1} a_k^{-(n+1-p)} s_0^{n-p-1}, \quad k = 1, 2 \end{gathered} \tag{2.6}$$

Integrating inequalities (2.6) with respect to ω from zero to the current value, we find

$$-A_1 s_0^{p-n+1}\omega \le f(s) \le -A_2 s_0^{p-n+1}\omega, \;\; f(s) \equiv (s^{p-n+1} - s_0^{p-n+1})/(p-n+1), \;\; s_0 = s(y_{i0}) \tag{2.7}$$

The quantities y_{i0} ($i = 1, 2, 3$) are defined by formula (2.3).

Since $f(s)$ is an increasing function, it follows from inequalities (2.7), regardless of the sign of the constant $p - n + 1$, that

$$\begin{gathered} s_0\varphi_1(\omega) \le s \le s_0\varphi_2(\omega) \\ \varphi_k(\omega) \equiv [1 - A_k(p-n+1)\omega]^{1/(p-n+1)}, \quad k = 1, 2 \end{gathered} \tag{2.8}$$

From relations (2.1) and (2.8), for s_c, we obtain

$$a_2^{-1} s_0 \varphi_1(\omega) \le s_c \le a_1^{-1} s_0 \varphi_2(\omega) \tag{2.9}$$

Then, from the last equation of (1.6) for $\dot{\omega}$, we shall have

$$t_{*2}^{-1}[\varphi_1(\omega)]^p \le (1-\omega)^m \dot{\omega} \le t_{*1}^{-1}[\varphi_2(\omega)]^p, \quad t_{*k}^{-1} \equiv B_2(a_k^{-1} s_0)^p, \quad k = 1, 2 \tag{2.10}$$

If $A_1(p - n + 1) < 1$, then the EPNI fractures after a finite time t_*. Actually, since, as a consequence of inequalities (2.8), the functions $[\varphi_k(\omega)]^{-p}$ ($k = 1.2$) are increasing functions, after integration of the first inequality we find from (2.10)

$$\frac{t_*}{t_{*2}} \le \int_0^1 [\varphi_1(\omega)]^{-p}(1-\omega)^m d\omega < [\varphi_1(1)]^{-p}\int_0^1 (1-\omega)^m d\omega = [\varphi_1(1)]^{-p}(m+1)^{-1} \tag{2.11}$$

If $A_1(p - n + 1) = 1$, then

$$\frac{t_*}{t_{*2}} \le \int_0^1 (1-\omega)^{m - A_1 p} d\omega \tag{2.12}$$

whence it follows that $t_* < \infty$ if $m + 1 - A_1 p > 0$.

If $A_2(p-n+1) > 1$, fracture will not occur since the value

$$\omega = \omega_0 \equiv A_2^{-1}(p-n+1)^{-1} < 1$$

will be attained after a time $t_0 \to \infty$ [4]. In fact, from inequalities (2.8) and (2.10) we obtain

$$\frac{t_0}{t_{*1}} \ge \int_0^{\omega_0} (1-\omega)^m [\varphi_2(\omega)]^{-p} d\omega \ge (1-\omega_0)^m \int_0^{\omega_0} \left(1-\frac{\omega}{\omega_0}\right)^{-p/(p-n+1)} d\omega =$$

$$= \frac{(1-\omega_0)^m}{A_2(n-1)} \lim_{\omega \to \omega_0 - 0} \left(1-\frac{\omega}{\omega_0}\right)^{-(n-1)(p-n+1)} = \infty$$

since $p-n+1 > A_2^{-1} > 0$ and $n-1 \ge 0$.

If $A_2(p-n+1) = 1$, then

$$\frac{t_*}{t_{*1}} \ge \int_0^1 (1-\omega)^{m-A_2 p} d\omega$$

and hence, when $m - A_2 p + 1 \le 0$, the time prior to fracture $t_* \to \infty$.

2. Suppose $n+1-p < 0$. In this case, the signs of all of the inequalities in formulae (2.5)–(2.8) change to the opposite signs and, instead of (2.9), we obtain

$$a_2^{-1} s_0 \varphi_2(\omega) \le s_c \le a_1^{-1} s_0 \varphi_1(\omega) \tag{2.13}$$

Hence, by repeating all the previous arguments, we can show that fracture will occur when $A_2(p-n+1) < 1$ and when $A_2 p = A_2(n-1) + 1 < m+1$, but there will be no fracture if $A_1(p-n+1) > 1$ or $A_1 p = A_1(n-1) + 1 \ge m+1$.

3. Suppose $n+1-p = 0$. Then, from equality (2.4), we find

$$s = s_0 \varphi(\omega), \quad \varphi(\omega) = (1-A_0\omega)^{1/2}; \quad A_0 = 2B_1 B_2^{-1} s_0^{-2} \tag{2.14}$$

and, from inequalities (2.1), we shall have

$$a_2^{-1} s_0 \varphi(\omega) \le s_c \le a_1^{-1} s_0 \varphi(\omega) \tag{2.15}$$

Hence it is clear that fracture will occur when $A_0 < 1$ and when $A_0 = 1 > p/2 - m$ but is impossible if $A_0 > 1$ or $A_0 = 1 \le p/2 - m$. These conclusions follow from the formulae presented above in which $p = n+1, n \ge 1$.

3. LOWER AND UPPER LIMITS OF THE TIME PRIOR TO FRACTURE

In order to find the exact values of the quantity t_* (that is, the times which have elapsed from the time when the loads ξ_i were applied at infinity up to the fracture of the EPNI), it is necessary to solve the non-linear system of equations (2.2) for $y_i = y_i(\omega)$ $(i = 1, 2, 3)$ with the initial conditions (2.3) and, after this, to substitute the function

$$s_c(\omega) = [\alpha_{ij}^c y_i(\omega) y_j(\omega)]^{1/2}$$

into the last equation of (1.6), whence it follows that

$$t_* = B_2^{-1} \int_0^1 [s_c(\omega)]^{-p} (1-\omega)^m d\omega \tag{3.1}$$

An exact solution of system (2.2) is possible in the simplest cases such as, for example, the case mentioned above when $s = a_0 s_c$ (a_0 = const) and the function $s_c = s_c(\omega)$ is determined from Eq. (2.4)

$$s_c = s_{c0}[1 - B_1 B_2^{-1} a_0^{-2}(p-n+1) s_{c0}^{n-p-1} \omega]^{1/(p-n+1)}, \quad s_{c0} = s_c(0) \tag{3.2}$$

However, using the inequalities presented in Section 2, it is possible to estimate the magnitude of t_* in all of the above cases when the sufficient conditions for fracture of the inclusion are satisfied. Integrating the second inequality of (2.10) and taking into account the fact that $[\varphi_k(\omega)]^{-p} > 1$ ($k = 1, 2$) when $\omega > 0$, we obtain a lower limit and, from inequalities (2.11) and (2.12), an upper limit in the following form

$$t_{*1}(m+1)^{-1} < t_* < t_{*2}[\varphi_1(1)]^{-p}(m+1)^{-1} \quad \text{when} \quad p-n < \min(1, A_1^{-1}-1)$$

$$t_{*1}(m+1)^{-1} < t_* < t_{*2}(m-A_1p+1)^{-1} \quad \text{when} \quad p-n < A_1^{-1}-1 < 1, \quad m-A_1p+1 > 0$$

In the case when $n + 1 - p < 0$, from inequalities (2.13) and the third equation of (1.6), we obtain equalities of the form (2.10) in which the functions $\varphi_1(\omega)$ and $\varphi_2(\omega)$ have to change places. Hence, we have the analogous estimates

$$t_{*1}(m+1)^{-1} < t_* < t_{*2}[\varphi_2(1)]^{-p}(m+1)^{-1} \quad \text{when} \quad 1 < p-n < A_2^{-1}-1$$

$$t_{*1}(m+1)^{-1} < t_* < t_{*2}(m-A_2p+1)^{-1} \quad \text{when} \quad p-n < A_2^{-1}-1 > 1, \quad m-A_2p+1 > 0$$

If $p = n + 1$, then, from relations (2.14) and (2.15) and the third equation of (1.6) for $\dot{\omega}$, we obtain inequalities of the form (2.10) in which it is necessary to put $\varphi_1(\omega) = \varphi_2(\omega) = \varphi(\omega)$. We shall therefore have

$$t_{*1}(m+1)^{-1} < t_* < t_{*2}[\varphi(1)]^{-p}(m+1)^{-1} \quad \text{when} \quad A_0 < 1$$

$$t_{*1}(m-p/2+1)^{-1} < t_* < t_{*2}(m-p/2+1)^{-1} \quad \text{when} \quad A_0 = 1 > p/2-m$$

The quantities $\varphi(\omega)$ and A_0 are defined by formulae (2.14).

4. AN INHOMOGENEOUS INCOMPRESSIBLE MEDIUM IN THE CASE OF PLANE STRAIN

We will now consider the case when the quadratic form $s_c^2 = \alpha_{ij}^c y_i y_j$ from relations (1.6) is positive semi-definite, that is, $\alpha_{ij}^c y_i y_j \geq 0$ when $y_i y_i \neq 0$. We will assume that the domain $S \cup S^*$ is isotropic, incompressible and is under conditions of plane strain so that $\kappa = 1$ in relations (1.1) and $s_e = \tau\mu^{*-1/2}$ and $s_c = \tau$ in the constitutive equations (1.6) for the EPNI, where $\tau \equiv [(y_1 - y_2)^2/4 + y_3^2]^{1/2}$ is the principal shear stress and μ^* is the shear modulus of the inclusion. Hence, when $y_1 = y_2 \neq 0$ and $y_3 = 0$, we have $\tau = 0$ and $A_{ij}y_iy_j > 0$, which follows from equalities (1.5) and (1.7). Consequently, the second equality of (2.1) cannot hold when $a_2 < \infty$. However, it is possible in this case also to obtain limits which are analogous to those presented in Sections 2 and 3.

For this purpose, we express Γ' and Γ from relations (1.4) when $\kappa = 1$ as

$$2\Gamma' = (1-m_0^2)B - \mu(1+m_0^2)\bar{D} - 2\mu m_0\bar{C}, \quad 2\Gamma = m_0(B+\mu\bar{D}) + A + \mu\bar{C} \tag{4.1}$$

Taking account of the fact that A and Γ are positive quantities and that $C = 2i\varepsilon^*$ (since $\varepsilon_{11}^* + \varepsilon_{22}^* = 0$) is pure imaginary, we find from the second equality of (4.1)

$$2\mu\varepsilon^* = m_0(\sigma_{12}^* - 2\mu\varepsilon_{12}^*) \tag{4.2}$$

It can be seen from equalities (4.1) and (4.2) that Γ' is independent of A. Substituting expression (4.2) into the first equality of (4.1) and using relations (1.5) and (1.6), after separating the real and imaginary parts we obtain

$$\begin{gathered} \xi_2 - \xi_1 = M_1^+(y_2 - y_1)/2 + 2M_2^+ f_2^c, \quad 2\xi_3 = M_1^- y_3 + M_2^- f_3^c \\ M_1^\pm = (1 \mp m_0^2) + (1 \pm m_0^2)\mu/\mu^*, \quad M_2^\pm = \mu(1 \pm m_0^2) \end{gathered} \tag{4.3}$$

where we have taken into account the fact that $f_1^c + f_2^c = 0$.

Since ξ_k = const (k = 1, 2, 3), then, differentiating equalities (4.3) with respect to ω, we obtain a system of the form of (2.2) in the two quantities: $y_2 - y_1$ and y_3 as functions of ω

$$M_1^+(y_2' - y_1')/2 + 2M_2^+B_1B_2^{-1}\tau^{n-p}\partial\tau/\partial y_2 = 0, \quad M_1^- y_3' + M_2^- B_1B_2^{-1}\tau^{n-p}\partial\tau/\partial y_3 = 0 \tag{4.4}$$

the initial data for which, that is, the values of $y_{20} - y_{10}$ and y_{30}, follow from equalities (4.3) when $f_2^c = f_3^c = 0$.

Multiplying the first equation of (4.4) by $(y_2 - y_1)/M_2^+$ and the second equation by y_3/M_2^- and adding them, we shall have the analogue of Eq. (2.4)

$$TT' + B_1B_2^{-1}\tau^{n-p+1} = 0 \tag{4.5}$$

$$T^2 \equiv a_1^2\left(\frac{y_2 - y_1}{2}\right)^2 + a_2^2 y_3^2, \quad a_1^2 = \frac{\delta}{\mu} + \frac{1}{\mu^*}, \quad a_2^2 = \frac{1}{\mu\delta} + \frac{1}{\mu^*} \quad (a_1 \le a_2)$$

$$\delta = \frac{1 - m_0^2}{1 + m_0^2} \quad (0 < \delta \le 1)$$

Both of the quadratic forms τ^2 and T^2 in $y_2 - y_1$ and y_3 are positive definite and an inequality of the form (2.1)

$$a_1\tau \le T \le a_2\tau$$

holds. Hence, all the arguments presented in Sections 2 and 3 still hold, where it is necessary to replace s by T and s_c by τ, while the quantities indicated above are taken as a_1 and a_2.

Note that, if $\xi_2 = \xi_1$ = const (or $\xi_3 = 0$), which, as can be seen from equalities (1.4), corresponds to the value $\alpha = \pi/4$ (or $\alpha = 0$ and $\alpha = \pi/2$), then, in view of relations (4.3), $y_{20} = y_{10}$ (or $y_{30} = 0$) and, as a consequence of system (4.4), it can easily be seen that the equalities $y_2 = y_1$ (or $y_3 = 0$) will be satisfied at any instant of time $t > 0$ and system (4.4) will degenerate into a single equation in y_3 (or $y_2 - y_1$). The solution of this equation has the form (3.2) since $s = a_0 s_c$, where $s_c = \tau$, $\tau = |y_3|$, $a_0 = a_2$ (or $\tau = |y_2 - y_1|/2$, $a_0 = a_1$), and a_1 and a_2 are defined by formulae (4.5). It has already been mentioned that this solution is analogous to that obtained previously in [4] for the case of a uniaxial stressed state and enables one, using expressions (3.1) and (3.2), to give a precise answer to the question concerning the possibility of the fracture of an EPNI and to find the time t_*. For example, if

$$A_3(p - n + 1) < 1, \quad A_3 \equiv B_1B_2^{-1}a_0^{-2}\tau_0^{n-p-1} \tag{4.6}$$

then $t_* < \infty$ [4]. In particular, condition (4.6) will be satisfied when $p - n + 1 \le 0$ regardless of the magnitude of A_3.

It is interesting to note the following. We assume that, at the instant of time $t = 0$, the (homogeneous) elastic strains F_{i0} are in instantaneous communication with the inclusion S^*. When $t > 0$, these elastic strains remain fixed, that is, $\dot{F}_i = 0$ and, by virtue of Eqs (1.6), a relaxation process will also occur, which is described by a system of equations obtained from (1.6) by differentiation with respect to t (or with respect to ω), taking account of the fact that $\dot{F}_i = 0$ (or $F_i' = 0$) (i = 1, 2, 3). (These conditions can be achieved by choosing the stresses ξ_i at infinity which, in the case of known y_i, are found from relations (1.5) when $F_i = F_{i0}$ (i = 1, 2, 3).) In the case of an isotropic, incompressible medium under plane strain, which is being considered here, we obtain Eq. (2.4) in which $s_c = \tau$, $s = \tau/\sqrt{\mu^*}$ (it is obtained from Eq. (4.5) when $\mu \to \infty$). It then follows from relations (3.2) that the time t_* will be finite or infinite depending on the sign of the quantity $1 - A_4(p - n + 1)$, where $A_4 = H_cH_e^{-1}$, $H_c = B_1B_2^{-1}\tau_0^{n-p}$, $H_c(f_2, f_3) \equiv 2(f_2^{c2} + f_3^{c2})^{1/2}$ is the principal elastic shear at the instant of fracture under conditions of creep when $\tau = \tau_0$ = const, and $H_e = \tau_0/\mu^*$ is the principal elastic shear when $\tau = \tau_0$. For real media $A_4 > 1$, and fracture is therefore only possible for brittle media for which $n > p$ since $t_* < \infty$ when $A_4(p - n + 1) < 1$, that is, $p - n < A_4^{-1} < 0$ [4]. When $\xi_3 = 0$ or $\xi_2 - \xi_1 = 0$ in the problem being considered here, condition (4.6) for the finiteness of the time t_*, that is,

$$\mu^{*-1}a_0^{-2}A_4(p - n + 1) < 1 \quad (a_0 = a_1 \quad \text{or} \quad a_0 = a_2)$$

can also be satisfied for elastic materials for which $n < p$ and $A_4 \gg 1$ [4]. In fact, the inequalities $0 < p - n < \mu^* a_0^2 A_4^{-1} - 1$ can hold, if $\mu^* a_0^2 A_4^{-1} > 1$, that is, (after substituting the values of a_1 and a_2

according to formulae (4.5) instead of a_0) when $\delta\mu^*\mu^{-1} > A_4 - 1$ and $\delta^{-1}\mu^*\mu^{-1} > A_4 - 1$ respectively. This is possible if the geometrical parameter of the EPNI and the moduli of elasticity of the medium and the inclusion satisfy the conditions: $\delta\mu^*\mu^{-1} \gg 1$ or $\delta^{-1}\mu^*\mu^{-1} \gg 1$. For example, the second inequality can be satisfied for any finite ratio $\mu^*\mu^{-1}$ owing to the choice of the small parameter $\delta\,(m_0 \to 1)$ when $\delta \to 0$, that is, the elliptic inclusion degenerates into a slit filled with a non-linear medium [2].

5. CONCLUDING REMARKS

The problems considered above concerning the possibility (or impossibility) of the fracture of an elliptic physically non-linear inclusion (EPNI), which is under conditions of stress relaxation, in an elastic plane under the action of constant loads (that is, under constant strains) at infinity, can be extended to the case of an inclusion S^* of arbitrary shape in a finite elastic (or viscoelastic) domain S with an external boundary L. In fact, we assume that, when $t = 0$, a homogeneous stress–strain state is created in the domain S^* with constitutive equations of the form (1.6) due to the external loads p_{k0} acting on L. When $t > 0$, the strains in S^* must remain fixed, that is, $\dot{F}_i = 0$ ($i = 1, 2, 3$), which causes stress relaxation process in the region S^*. This can also be achieved by choosing the external forces $p_k = p_k(t)$ on L, if the exact solution of system (1.6) is known when $F_i = F_{i0}$. The problem of finding the functions $p_k = p_k(t)$ in L in the case of a known homogeneous stress–strain state in S^*, that is, the specified functions $F_i = F_i(t)$ and $y_i = y_i(t)$ ($i = 1, 2, 3$), has been considered previously [1, 5].

The above-mentioned exact solution can be obtained, for example, in the case of an inhomogeneous, incompressible medium under plane strain considered in Section 4. Actually, from equalities (2.4) when $s_c = \tau$ and $s = \tau/\sqrt{\mu^*}$, we find a function $\tau = \tau(\omega)$ of the form of (3.2), where $a_0^{-2} = \mu^*$. Then, from the system, which follows from (1.6) after differentiation with respect to ω, and the equalities $F_i' = 0$ ($i = 2, 3$), we determine $y_2 - y_1$ and y_3 as functions of ω:

$$\tau = \tau_0\Phi(\omega), \quad y_2 - y_1 = (y_{20} - y_{10})\Phi(\omega), \quad y_3 = y_{30}\Phi(\omega)$$
$$\Phi(\omega) \equiv [1 - B_1 B_2^{-1}\mu^*(p - n + 1)\tau_0^{n-p-1}\omega]^{1/(p-n+1)} \tag{5.1}$$

From the last equation of (1.6), we obtain

$$t(\omega) = B_2^{-1}\tau_0^{-p}\int_0^{\omega}[\Phi(\omega)]^{-p}(1-\omega)^m d\omega \tag{5.2}$$

The inverse function $\omega = \omega(t)$ can be found (numerically, for example) from relation (5.2), and, on substituting this into the second and third equations of (5.1), we get $y_2 - y_1$ and y_3 as functions of t.

Note that the quantity $A = (y_1 + y_2)/2$ has no effect on the initial stress–strain state in the domain S^* and the subsequent relaxation process. It can therefore be chosen arbitrarily.

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